Algebraic independence of functions satisfying certain Mahler type functional equations and its applications

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1 Introduction and results.

One of the techniques used to prove the algebraic independence of numbers is Mahler's method, which deals with the values of so-called Mahler functions satisfying a certain type of functional equation. In order to apply the method, one must confirm the algebraic independence of the Mahler functions themselves. This can be reduced, in many cases, to their linear independence modulo the rational function field, that is, the problem of determining whether a nonzero linear combination of them is a rational function or not. In the case of one variable, this can be treated by arguments involving poles of rational functions. However, in the case of several variables, this method is not available. In this paper we shall resolve this problem by considering a generic point of an irreducible algebraic variety. Theorems 1 and 2 in this paper assert that certain types of functional equations in several variables have no nontrivial rational function solutions. As applications, we shall prove the algebraic independence of various kinds of reciprocal sums of linear recurrences in Theorems 3, 4, and 5, and that of the values at algebraic numbers of power series, Lambert series, and infinite products generated by linear recurrences in Theorem 6.

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. If $\boldsymbol{z} = (z_1, \ldots, z_n)$ is a point of $\boldsymbol{C}^n$ with $\boldsymbol{C}$ the set of complex numbers, we define a transformation $\Omega : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right). \tag{1}$$

Let $\{a_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying

$$a_{k+n} = c_1 a_{k+n-1} + \cdots + c_n a_k \quad (k = 0, 1, 2, \ldots), \tag{2}$$

where $a_0, \ldots, a_{n-1}$ are not all zero and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. We define a polynomial associated with (2) by

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n. \tag{3}$$

In this paper, we always assume that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity and that $\{a_k\}_{k\geq 0}$ is not a geometric progression unless otherwise mentioned. We define a monomial

$$P(\boldsymbol{z}) = z_1^{a_{n-1}} \cdots z_n^{a_0}, \tag{4}$$

which is denoted similarly to (1) by

$$P(\boldsymbol{z}) = (a_{n-1}, \ldots, a_0)\boldsymbol{z}. \tag{5}$$

Let

$$\Omega = \begin{pmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \ldots & \ldots & 0 \end{pmatrix}. \tag{6}$$

It follows from (1), (2), and (5) that

$$\begin{aligned} P(\Omega^k \boldsymbol{z}) &= (a_{n-1}, \ldots, a_0)\Omega^k \boldsymbol{z} \\ &= (a_{k+n-1}, \ldots, a_k)\boldsymbol{z} \\ &= z_1^{a_{k+n-1}} \cdots z_n^{a_k} \quad (k \geq 0). \end{aligned}$$

Let $F(z_1, \ldots, z_n)$ and $F[[z_1, \ldots, z_n]]$ denote the field of rational functions and the ring of formal power series in variables $z_1, \ldots, z_n$ with coefficients in a field F, respectively, and $F^\times$ the multiplicative group of nonzero elements of F. Throughout this paper, we denote by C a field of characteristic 0. The following are the main theorems of the present paper.

Theorem 1. *Suppose that $G(\boldsymbol{z}) \in C[[z_1, \ldots, z_n]]$ satisfies the functional equation of the form*

$$G(\boldsymbol{z}) = \alpha G(\Omega^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} Q_k(P(\Omega^k \boldsymbol{z})), \tag{7}$$

where $\alpha \neq 0$ *is an element of* C, Ω *is defined by* (6), $p > 0$, $q \geq 0$ *are integers, and* $Q_k(X) \in C(X)$ $(q \leq k \leq p+q-1)$ *are defined at* $X = 0$. *If* $G(\boldsymbol{z}) \in C(z_1, \ldots, z_n)$, *then* $G(\boldsymbol{z}) \in C$ *and* $Q_k(X) \in C$ $(q \leq k \leq p+q-1)$.

Theorem 2. *Suppose that* $G(\boldsymbol{z})$ *is a nonzero element of the quotient field of* $C[[z_1, \ldots, z_n]]$ *satisfying the functional equation of the form*

$$G(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} Q_k(P(\Omega^k \boldsymbol{z})) \right) G(\Omega^p \boldsymbol{z}), \tag{8}$$

where Ω, p, q, *and* $Q_k(X)$ *are as in Theorem* 1. *Assume that* $Q_k(0) \neq 0$. *If* $G(\boldsymbol{z}) \in C(z_1, \ldots, z_n)$, *then* $G(\boldsymbol{z}) \in C$ *and* $Q_k(X) \in C^{\times}$ $(q \leq k \leq p+q-1)$.

First we shall state our results on algebraic independence of reciprocal sums of linear recurrences, Theorems 3, 4, and 5, obtained as applications of Theorem 1. We prepare some notations.

Let $\{R_k\}_{k\geq 0}$ be a linear recurrence expressed as

$$R_k = g_1 \rho_1^k + \cdots + g_r \rho_r^k \quad (k \geq 0), \tag{9}$$

where $g_1, \ldots, g_r$ are nonzero algebraic numbers and $\rho_1, \ldots, \rho_r$ are nonzero distinct algebraic numbers satisfying

$$|\rho_1| > \max\{1, |\rho_2|, \ldots, |\rho_r|\}. \tag{10}$$

Typical examples of such $\{R_k\}_{k\geq 0}$ are the Fibonacci numbers $\{F_k\}_{k\geq 0}$ defined by

$$F_0 = 0, \ F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0)$$

and the Lucas numbers $\{L_k\}_{k\geq 0}$ defined by

$$L_0 = 2, \ L_1 = 1, \quad L_{k+2} = L_{k+1} + L_k \quad (k \geq 0),$$

since

$$F_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right) \quad (k \geq 0)$$

and

$$L_k = \left(\frac{1+\sqrt{5}}{2} \right)^k + \left(\frac{1-\sqrt{5}}{2} \right)^k \quad (k \geq 0).$$

We shall prove the algebraic independence of reciprocal sums of linear recurrences such as

$$\sum_{k\geq 0}{}' \frac{b_k}{(R_{a_k+h})^m}, \tag{11}$$

where $\{b_k\}_{k\geq 0}$ is a linear recurrence of algebraic numbers not identically zero, $\{a_k\}_{k\geq 0}$ is as above, and $m \geq 1,\ h$ are integers. Here and in what follows, the sum $\sum'_{k\geq 0}$ is taken over those k which satisfy $a_k + h \geq 0$ and $R_{a_k+h} \neq 0$. For example, the algebraic independence of the numbers

$$\sum_{k\geq 0}{}' \frac{1}{(F_{F_k+h})^m} \qquad (h \in \boldsymbol{Z},\ m \in \boldsymbol{N}^+)$$

can be deduced from Theorem 4 below. Here $\boldsymbol{Z}$ and $\boldsymbol{N}^+$ denote the sets of rational and positive integers, respectively.

It is interesting to compare our results to those obtained by various authors in the case where $\{a_k\}_{k\geq 0}$ is a geometric progression. Lucas [7] showed that

$$\sum_{k\geq 0} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2}.$$

Let $\{s_k\}_{k\geq 0}$ be a periodic sequence of algebraic numbers not identically zero. Bundschuh and Pethö [3] proved by Mahler's method that

$$\sum_{k\geq 0} \frac{s_k}{F_{2^k}}$$

is transcendental if $\{s_k\}_{k\geq 0}$ is not a constant sequence and that

$$\sum_{k\geq 0} \frac{s_k}{L_{2^k}}$$

is transcendental for any $\{s_k\}_{k\geq 0}$. Let $c \geq 1$ and d be integers. Recently, Nishioka, Tanaka, and Toshimitsu [13] proved that if $\{s_k\}_{k\geq 0}$ is not a constant sequence, the numbers

$$\sum_{k\geq 0}{}' \frac{s_k}{(F_{cd^k+h})^m} \qquad (d \geq 2,\ h \in \boldsymbol{Z},\ m \in \boldsymbol{N}^+) \tag{12}$$

are algebraically independent, and if $\{s_k\}_{k\geq 0}$ is a constant sequence, the numbers (12) excepting the algebraic number $\sum'_{k\geq 0} s_k/F_{c2^k}$ are algebraically independent; and also the numbers

$$\sum_{k\geq 0}{}' \frac{s_k}{(L_{cd^k+h})^m} \qquad (d \geq 2,\ h \in \boldsymbol{Z},\ m \in \boldsymbol{N}^+)$$

are algebraically independent for any $\{s_k\}_{k\geq 0}$. These results depend on the fact that the recurrences $\{F_k\}_{k\geq 0}$ and $\{L_k\}_{k\geq 0}$ are binary, namely these can be expressed as (9) with $r = 2$. In the case of $m = 1$, the transcendence of each of these numbers has already been proved by Becker and Töpfer [1]. For a general $\{R_k\}_{k\geq 0}$ not necessarily binary, only the transcendency result has been obtained also by Becker and Töpfer [1] : If $\rho_1, \ldots, \rho_r$ satisfying (10) are multiplicatively independent, then the number

$$\sum_{k\geq 0}{}' \frac{s_k}{R_{cd^k}}$$

is transcendental (cf. Remark 2 below).

Our results are concerned with the algebraic independence of the numbers (11) with $\{a_k\}_{k\geq 0}$ not a geometric progression. It is not necessary in our results to assume that $\rho_1, \ldots, \rho_r$ are multiplicatively independent. In what follows, $\boldsymbol{N}$ denotes the set of nonnegative integers and $\overline{\boldsymbol{Q}}$ the field of algebraic numbers.

Theorem 3. *Let $\{R_k\}_{k\geq 0}$ be a linear recurrence represented as* (9) *with* (10). *Then the numbers*

$$\sum_{k\geq 0}{}' \frac{k^l \alpha^k}{(R_{a_k})^m} \qquad (\alpha \in \overline{\boldsymbol{Q}}^{\times},\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^+) \tag{13}$$

are algebraically independent.

Theorem 3 implies the algebraic independence of the numbers

$$\sum_{k\geq 0}{}' \frac{b_k}{(R_{a_k})^m} \qquad (m \in \boldsymbol{N}^+),$$

since a linear recurrence $\{b_k\}_{k\geq 0}$ of algebraic numbers not identically zero can be expressed as the linear combination of the sequences $\{k^l \alpha^k\}_{k\geq 0}$ $(\alpha \in \overline{\boldsymbol{Q}}^{\times},\ l \in \boldsymbol{N})$ with algebraic coefficients.

Remark 1. It is proved by the author [14, Remark 4] that

$$a_k = b\gamma^k + o(\gamma^k),$$

where $\gamma > 1$ and $b > 0$, so that by (10) each sum in (13) converges.

Remark 2. It is still open to prove the algebraic independence of the numbers (13) with $\{a_k\}_{k\geq 0}$ a geometric progression and without the assumption that $\rho_1, \ldots, \rho_r$ are multiplicatively independent.

Corollary 1. *In addition to the assumptions on* $\Phi(X)$, *suppose that* $\Phi(X)$ *has only simple roots. Then the numbers*

$$\sum_{k\geq 0}{}' \frac{k^l\alpha^k}{(a_{a_k})^m} \qquad (\alpha \in \overline{\boldsymbol{Q}}^{\times},\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^+)$$

are algebraically independent.

Proof. Since $\Phi(X)$ has only simple roots, a_k in place of R_k can be expressed as (9) with distinct roots $\rho_1, \ldots, \rho_r$ of $\Phi(X)$. And (10) is also satisfied by the condition on $\Phi(X)$ (see Tanaka [14, Proof of Lemma 4]). Thus we can take a_k as R_k.

Example. Let $\{T_k\}_{k\geq 0}$ be so-called "Tribonacci" numbers defined by

$$T_{k+3} = T_{k+2} + T_{k+1} + T_k \quad (k = 0, 1, 2, \ldots)$$

with the initial values $T_0 = 0,\ T_1 = 1$, and $T_2 = 2$ and let $\{b_k\}_{k\geq 0}$ be a linear recurrence of algebraic numbers not identically zero. Then the numbers

$$\sum_{k\geq 1} \frac{b_k}{(T_{T_k})^m} \qquad (m \in \boldsymbol{N}^+)$$

are algebraically independent. We remark that T_k can be expressed as (9) with $r = 3$ and ρ_1, ρ_2, ρ_3 satisfying $\rho_1\rho_2\rho_3 = 1$, so that ρ_1, ρ_2, and ρ_3 are multiplicatively dependent.

If $\{R_k\}_{k\geq 0}$ is binary, we can deduce from Theorem 1 the algebraic independence of the numbers (11) for various $h \in \boldsymbol{Z}$, as in the case where $\{a_k\}_{k\geq 0}$ is a geometric progression stated above.

Theorem 4. *Let* $\{R_k\}_{k\geq 0}$ *be a binary recurrence represented as*

$$R_k = g_1\rho_1^k + g_2\rho_2^k \quad (k \geq 0),$$

where g_1, g_2, ρ_1, *and* ρ_2 *are nonzero algebraic numbers satisfying* $|\rho_1| > \max\{1, |\rho_2|\}$. *Then the numbers*

$$\sum_{k\geq 0}{}' \frac{k^l\alpha^k}{(R_{a_k+h})^m} \qquad (\alpha \in \overline{\boldsymbol{Q}}^{\times},\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^+,\ h \in \boldsymbol{Z}) \tag{14}$$

are algebraically independent.

Theorem 5. *Let* $\{R_k\}_{k\geq 0}$ *be as in Theorem* 4. *Then the numbers*

$$\sum_{k\geq 0}{}' \frac{k^l \alpha^k}{R_{ma_k+h}} \qquad (\alpha \in \overline{\boldsymbol{Q}}^{\times},\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^{+},\ h \in \boldsymbol{Z}) \tag{15}$$

are algebraically independent.

Corollary 2. *Let* $\{R_k\}_{k\geq 0}$ *be a binary recurrence defined by*

$$R_{k+2} = A_1 R_{k+1} + A_2 R_k \quad (k \geq 0),$$

where A_1 *and* A_2 *are real algebraic numbers satisfying* $A_1 \neq 0, |A_2| \geq 1,$ *and* $\Delta := A_1^2 + 4A_2 > 0$. *Suppose that* $\{R_k\}_{k\geq 0}$ *is not a geometric progression. Then the numbers* (14) *or* (15) *are algebraically independent.*

Example. Let $\{F_k\}_{k\geq 0}$ be the Fibonacci numbers and let $\{b_k\}_{k\geq 0}$ be a linear recurrence of algebraic numbers not identically zero. Then the numbers

$$\sum_{k\geq 0}{}' \frac{b_k}{(F_{F_k+h})^m} \qquad (h \in \boldsymbol{Z},\ m \in \boldsymbol{N}^{+})$$

are algebraically independent and so are the numbers

$$\sum_{k\geq 0}{}' \frac{b_k}{F_{mF_k+h}} \qquad (h \in \boldsymbol{Z},\ m \in \boldsymbol{N}^{+}).$$

Remark 3. In the case where $\{a_k\}_{k\geq 0}$ is a geometric progression, a similar result to Corollary 2 is obtained by Nishioka [12] under the assumption that R_0, R_1, A_1, and A_2 are rational integers and $m = 1$.

Next we state an application of Theorem 1 as well as Theorem 2.

Theorem 6. *Let* $\{a_k\}_{k\geq 0}$ *be a linear recurrence satisfying* (2) *with positive initial values* $a_0, \ldots, a_{n-1}$. *Let* $\alpha_1, \ldots, \alpha_r$ *be algebraic numbers with* $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$ *such that none of* α_i/α_j $(1 \leq i < j \leq r)$ *is a root of unity. Then the numbers*

$$\sum_{k\geq 0} \alpha_i^{a_k}, \quad \sum_{k\geq 0} \frac{\alpha_i^{a_k}}{1-\alpha_i^{a_k}}, \quad \prod_{k\geq 0} (1-\alpha_i^{a_k}) \qquad (1 \leq i \leq r)$$

are algebraically independent.

REMARK 4. The assumption that none of α_i/α_j $(1 \leq i < j \leq r)$ is a root of unity cannot be removed. For example, suppose that the initial values $a_0, \ldots, a_{n-1}$ are divided by an integer d (> 1). Then by the linear recurrence relation (2), a_k is divided by d for any $k \geq 0$. If α_i/α_j is a d-th root of unity for some distinct i and j, then $\alpha_i^{a_k} = \alpha_j^{a_k}$ $(k \geq 0)$ and so the numbers considered in Theorem 6 are algebraically dependent. Even in the case where $a_0, \ldots, a_{n-1}$ have no common factor, the assumption is also inevitable as the following example shows:

Let $\{a_k\}_{k\geq 0}$ be a linear recurrence defined by

$$a_0 = 2,\ a_1 = 3,\quad a_{k+2} = 6a_{k+1} + a_k \quad (k = 0, 1, 2, \ldots).$$

We put

$$f(z) = \sum_{k\geq 0} z^{a_k}, \quad g(z) = \sum_{k\geq 0} \frac{z^{a_k}}{1 - z^{a_k}}, \quad h(z) = \prod_{k\geq 0} (1 - z^{a_k}).$$

Let α be any algebraic number with $0 < |\alpha| < 1$ and $\zeta = e^{\pi\sqrt{-1}/3} = (1 + \sqrt{-3})/2$. Then

$$2f(\alpha) + f(\zeta\alpha) - f(\zeta^2\alpha) - 2f(\zeta^3\alpha) - f(\zeta^4\alpha) + f(\zeta^5\alpha) = 0,$$
$$2g(\alpha) + g(\zeta\alpha) - g(\zeta^2\alpha) - 2g(\zeta^3\alpha) - g(\zeta^4\alpha) + g(\zeta^5\alpha) = 0,$$

and

$$h(\alpha)^2 h(\zeta\alpha) h(\zeta^2\alpha)^{-1} h(\zeta^3\alpha)^{-2} h(\zeta^4\alpha)^{-1} h(\zeta^5\alpha) = 1,$$

since $a_{2k} \equiv 2 \pmod 6$ and $a_{2k+1} \equiv 3 \pmod 6$ for any $k \geq 0$.

REMARK 5. The author [14] obtained the necessary and sufficient condition for the numbers $\sum_{k\geq 0} \alpha_1^{a_k}, \ldots, \sum_{k\geq 0} \alpha_r^{a_k}$ in Theorem 6 to be algebraically dependent: Let $\{a_k\}_{k\geq 0}$ be a linear recurrence satisfying (2). Define $f(z) = \sum_{k\geq 0} z^{a_k}$ and let $\alpha_1, \cdots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leq i \leq r)$. Then the following three properties are equivalent:

(i) $f(\alpha_1), \ldots, f(\alpha_r)$ are algebraically dependent.

(ii) $1, f(\alpha_1), \ldots, f(\alpha_r)$ are linearly dependent over $\overline{\boldsymbol{Q}}$.

(iii) There exist a non-empty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ of $\{\alpha_1, \ldots, \alpha_r\}$, roots of unity $\zeta_1, \ldots, \zeta_s$, an algebraic number γ with $\alpha_{i_q} = \zeta_q \gamma$ $(1 \le q \le s)$, and algebraic numbers $\xi_1, \ldots, \xi_s$, not all zero, such that

$$\sum_{q=1}^{s} \xi_q {\zeta_q}^{a_k} = 0$$

for all sufficiently large k.

Remark 6. If $\{a_k\}_{k\ge 0}$ is a geometric progression, namely $a_k = cd^k$ $(k \ge 0)$ for some integers $c \ge 1$ and $d \ge 2$, each of the numbers in Theorem 6 is transcendental by the theorem of Mahler [8] ; however Theorem 6 is not valid in this case. Indeed, let

$$f(z) = \sum_{k\ge 0} z^{cd^k}, \quad g(z) = \sum_{k\ge 0} \frac{z^{cd^k}}{1 - z^{cd^k}}, \quad h(z) = \prod_{k\ge 0} (1 - z^{cd^k}).$$

Let α be any algebraic number with $0 < |\alpha| < 1$. We put $\alpha_1 = \alpha,\ \alpha_2 = \alpha^d,\ r = 2$, so that α_1/α_2 is not a root of unity. Then we have

$$f(\alpha_1) - f(\alpha_2) = \alpha^c,\ g(\alpha_1) - g(\alpha_2) = \frac{\alpha^c}{1-\alpha^c},\ \frac{h(\alpha_1)}{h(\alpha_2)} = 1 - \alpha^c \ \in \ \overline{\boldsymbol{Q}}.$$

Remark 7. The power series expansions of some of infinite products in Theorem 6 have interesting property. Beresin, Levine, and Lubell [2] proved that if

$$\prod_{k\ge 0} (1 - z^{F_{k+2}}) = \sum_{k\ge 0} \epsilon(k) z^k,$$

where $\{F_k\}_{k\ge 0}$ is the Fibonacci numbers, then $\epsilon(k) = 0$ or ± 1 for any $k \ge 0$.

2 Proofs of Theorems 3–6.

In this section we derive Theorems 3, 4, 5, and 6 from Theorems 1 and 2 by using Lemmas 1–5 below. Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue (cf. Gantmacher [4, p. 66, Theorem 3]). We suppose that Ω and a point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties:

(I) Ω is non-singular and none of its eigenvalues is a root of unity, so that in particular $\rho > 1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $\Omega^k \boldsymbol{\alpha} = (\alpha_1^{(k)}, \dots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}| \le -c\rho^k \quad (1 \le i \le n)$$

for all sufficiently large k, where c is a positive constant.

(IV) For any nonzero $f(\boldsymbol{z}) \in \boldsymbol{C}[[z_1, \dots, z_n]]$ which converges in some neighborhood of the origin, there are infinitely many $k \in \boldsymbol{N}^+$ such that $f(\Omega^k \boldsymbol{\alpha}) \ne 0$.

We note that the property (II) is satisfied if every eigenvalue of Ω of absolute value ρ is a simple root of the minimal polynomial of Ω.

Lemma 1 (Tanaka [14, Lemma 4, Proof of Theorem 2])**.** *Suppose that* $\Phi(\pm 1) \ne 0$ *and the ratio of any pair of distinct roots of* $\Phi(X)$ *is not a root of unity, where* $\Phi(X)$ *is the polynomial defined by* (3)*. Let* Ω *be the matrix defined by* (6) *and* $\beta_1, \dots, \beta_s$ *multiplicatively independent algebraic numbers with* $0 < |\beta_j| < 1$ $(1 \le j \le s)$*. Let* p *be a positive integer and put*

$$\Omega' = \mathrm{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{s}).$$

Then the matrix Ω' *and the point*

$$\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s)$$

have the properties (I)$-$(IV)*.*

Lemma 2 (Nishioka [9])**.** *Let* K *be an algebraic number field. Suppose that* $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}) \in K[[z_1, \dots, z_n]]$ *converge in an* n*-polydisc* U *around the origin and satisfy the functional equation of the form*

$$\begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega\boldsymbol{z}) \\ \vdots \\ f_m(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{z}) \\ \vdots \\ b_m(\boldsymbol{z}) \end{pmatrix}, \tag{16}$$

where A is an $m \times m$ matrix with entries in K and $b_i(\boldsymbol{z}) \in K(z_1, \ldots, z_n)$. Assume that the $n \times n$ matrix Ω and a point $\boldsymbol{\alpha} \in U$ whose components are nonzero algebraic numbers have the properties (I)$-$(IV)*. If $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ are algebraically independent over $K(z_1, \ldots, z_n)$, then $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$ are algebraically independent.*

Lemma 3 (Kubota [5], see also Nishioka [11])**.** *Let K be an algebraic number field. Suppose that $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in K[[z_1, \ldots, z_n]]$ converge in an n-polydisc U around the origin and satisfy the functional equations*

$$f_i(\Omega \boldsymbol{z}) = a_i(\boldsymbol{z}) f_i(\boldsymbol{z}) + b_i(\boldsymbol{z}) \quad (1 \leq i \leq m),$$

where $a_i(\boldsymbol{z}), b_i(\boldsymbol{z}) \in K(z_1, \ldots, z_n)$ and $a_i(\boldsymbol{z})$ are defined and nonzero at the origin. Assume that the $n \times n$ matrix Ω and a point $\boldsymbol{\alpha} \in U$ whose components are nonzero algebraic numbers have the properties (I)$-$(IV) *and that $a_i(\boldsymbol{z})$ are defined and nonzero at $\Omega^k \boldsymbol{\alpha}$ for all $k \geq 0$. If $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ are algebraically independent over $K(z_1, \ldots, z_n)$, then $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$ are algebraically independent.*

Lemma 3 is essentially due to Kubota [5] and improved by Nishioka [11].

Let $L = C(z_1, \ldots, z_n)$ and let M be the quotient field of $C[[z_1, \ldots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries having the property (I). We define an endomorphism $\tau : M \to M$ by

$$f^\tau(\boldsymbol{z}) = f(\Omega \boldsymbol{z}) \quad (f(\boldsymbol{z}) \in M) \tag{17}$$

and a subgroup H of $L^\times$ by

$$H = \{\ g^\tau g^{-1} \mid g \in L^\times\ \}.$$

Lemma 4 (Nishioka [9])**.** *Suppose that $f_{ij} \in M$ $(i = 1, \ldots, k,\ j = 1, \ldots, n(i))$ satisfy the functional equation of the form*

$$\begin{pmatrix} f_{i\,1} \\ \vdots \\ \vdots \\ f_{i\,n(i)} \end{pmatrix} = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ a^{(i)}_{2\,1} & a_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a^{(i)}_{n(i)\,1} & \cdots & a^{(i)}_{n(i)\,n(i)-1} & a_i \end{pmatrix} \begin{pmatrix} f^\tau_{i\,1} \\ \vdots \\ \vdots \\ f^\tau_{i\,n(i)} \end{pmatrix} + \begin{pmatrix} b_{i\,1} \\ \vdots \\ \vdots \\ b_{i\,n(i)} \end{pmatrix},$$

where $a_i,\ a_{st}^{(i)} \in C,\ a_i \neq 0,\ a_{s\,s-1}^{(i)} \neq 0,$ *and* $b_{ij} \in L$*. If* f_{ij} $(i = 1, \dots, k,\ j = 1, \dots, n(i))$ *are algebraically dependent over* L*, then there exist a non-empty subset* $\{i_1, \dots, i_r\}$ *of* $\{1, \dots, k\}$ *and nonzero elements* $c_1, \dots, c_r$ *of* C *such that*

$$a_{i_1} = \cdots = a_{i_r}, \quad c_1 f_{i_1 1} + \cdots + c_r f_{i_r 1} \in L.$$

Lemma 5 (Kubota [5], see also Nishioka [11])**.** *Let* $f_i \in M$ $(i = 1, \dots, h)$ *satisfy*

$$f_i^\tau = a f_i + b_i,$$

where $a \in L^\times$ *and* $b_i \in L$ $(1 \leq i \leq h)$*, and let* $f_i \in M^\times$ $(i = h+1, \dots, m)$ *satisfy*

$$f_i^\tau = a_i f_i,$$

where $a_i \in L^\times$ $(h+1 \leq i \leq m)$*. Suppose that* $a,\ a_i$*, and* b_i *have the following properties*:

(i) *If* $c_i \in C$ $(1 \leq i \leq h)$ *are not all zero, there is no element* g *of* L *such that*

$$ag - g^\tau = \sum_{i=1}^{h} c_i b_i.$$

(ii) $a_{h+1}, \dots, a_m$ *are multiplicatively independent modulo* H*.*

Then the functions f_i $(1 \leq i \leq m)$ *are algebraically independent over* L*.*

Proof of Theorem 3. Let $\rho_1, \dots, \rho_r$ be the algebraic numbers in (9). There exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_s$ with $0 < |\beta_j| < 1$ $(1 \leq j \leq s)$ such that

$$\rho_1^{-1} = \zeta_1 \prod_{j=1}^{s} \beta_j^{e_{1j}}, \quad \rho_1^{-1} \rho_i = \zeta_i \prod_{j=1}^{s} \beta_j^{e_{ij}} \quad (2 \leq i \leq r), \tag{18}$$

where $\zeta_1, \dots, \zeta_r$ are roots of unity and e_{ij} $(1 \leq i \leq r,\ 1 \leq j \leq s)$ are nonnegative integers (cf. Loxton and van der Poorten [6], Nishioka [11]). Take a positive integer N such that $\zeta_i^N = 1$ for any i $(1 \leq i \leq r)$. We can choose a positive integer p and a nonnegative integer k_0 such that $a_{k+p} \equiv a_k \pmod{N}$ for any $k \geq k_0$. By Remark 1, there exists a nonnegative integer k_1 such that $a_{k+1} > a_k$ for all $k \geq k_1$. Therefore by (9) and (10), there exists a nonnegative integer $q \geq \max\{k_0, k_1\}$ such that $R_{a_k} \neq 0$ for all $k \geq q$. Let

$y_{j\lambda}$ $(1 \le j \le s,\ 1 \le \lambda \le n)$ be variables and let $\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn})$ $(1 \le j \le s)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_s)$. Define

$$f_m(x, \boldsymbol{y}) = \sum_{k \ge q} x^k \left(\frac{\zeta_1^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{1j}}}{g_1 + \sum_{i=2}^r g_i \zeta_i^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}} \right)^m \quad (m \ge 1),$$

where $P(\boldsymbol{z}),\ \boldsymbol{z} = (z_1, \dots, z_n)$, is the monomial given by (4) and Ω is the matrix given by (6). Letting

$$D = x\frac{\partial}{\partial x},\ \ \alpha \in \overline{\boldsymbol{Q}}^\times,\ \text{ and } \boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s),$$

we see that

$$D^l f_m(\alpha, \boldsymbol{\beta}) = \sum_{k \ge q} k^l \alpha^k \left(\frac{\rho_1^{-a_k}}{g_1 + \sum_{i=2}^r g_i (\rho_1^{-1}\rho_i)^{a_k}} \right)^m = \sum_{k \ge q} \frac{k^l \alpha^k}{(R_{a_k})^m}.$$

Hence

$$\sum_{k \ge 0}{}' \frac{k^l \alpha^k}{(R_{a_k})^m} - D^l f_m(\alpha, \boldsymbol{\beta}) \in \overline{\boldsymbol{Q}} \quad (\alpha \in \overline{\boldsymbol{Q}}^\times,\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^+),$$

and so it suffices to prove the algebraic independence of the values

$$D^l f_m(\alpha, \boldsymbol{\beta}) \quad (\alpha \in \overline{\boldsymbol{Q}}^\times,\ l \in \boldsymbol{N},\ m \in \boldsymbol{N}^+).$$

Let

$$\Omega' = \mathrm{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{s}).$$

Then $f_m(x, \boldsymbol{y})$ satisfies the functional equation

$$\begin{aligned} f_m(x, \boldsymbol{y}) \ &= \ x^p f_m(x, \Omega' \boldsymbol{y}) \\ &\quad + \sum_{k=q}^{p+q-1} x^k \left(\frac{\zeta_1^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{1j}}}{g_1 + \sum_{i=2}^r g_i \zeta_i^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}} \right)^m, \end{aligned} \tag{19}$$

where $\Omega' \boldsymbol{y} = (\Omega^p \boldsymbol{y}_1, \dots, \Omega^p \boldsymbol{y}_s)$, and so $D^l f_m(x, \boldsymbol{y})$ $(l \ge 1)$ satisfy

$$\begin{aligned} & D^l f_m(x, \boldsymbol{y}) \\ = \ & \sum_{\mu=0}^{l} \binom{l}{\mu} p^{l-\mu} x^p D^\mu f_m(x, \Omega' \boldsymbol{y}) \\ & + \sum_{k=q}^{p+q-1} k^l x^k \left(\frac{\zeta_1^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{1j}}}{g_1 + \sum_{i=2}^r g_i \zeta_i^{a_k} \prod_{j=1}^s P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}} \right)^m. \end{aligned} \tag{20}$$

We assume that the values $D^l f_m(\alpha_\sigma, \boldsymbol{\beta})$ $(0 \le l \le L,\ 1 \le m \le M,\ 1 \le \sigma \le t)$ are algebraically dependent, where $\alpha_1, \dots, \alpha_t$ are nonzero distinct algebraic numbers. It follows from (19) and (20) that $D^l f_m(\alpha_\sigma, \boldsymbol{y})$ $(0 \le l \le L,\ 1 \le m \le M,\ 1 \le \sigma \le t)$ satisfy the functional equation of the form (16), so that they are algebraically dependent over $\overline{\boldsymbol{Q}}(\boldsymbol{y})$ by Lemmas 1 and 2. Hence we see by Lemma 4 that

$$\alpha_1^p = \cdots = \alpha_\nu^p \tag{21}$$

and $f_m(\alpha_\sigma, \boldsymbol{y})$ $(1 \le m \le M,\ 1 \le \sigma \le \nu)$ are linearly dependent over $\overline{\boldsymbol{Q}}$ modulo $\overline{\boldsymbol{Q}}(\boldsymbol{y})$, changing the indices σ $(1 \le \sigma \le t)$ if necessary. Thus there are algebraic numbers $c_{m\sigma}$ $(1 \le m \le M,\ 1 \le \sigma \le \nu)$, not all zero, such that

$$F(\boldsymbol{y}) := \sum_{m=1}^{M} \sum_{\sigma=1}^{\nu} c_{m\sigma} f_m(\alpha_\sigma, \boldsymbol{y}) \in \overline{\boldsymbol{Q}}(\boldsymbol{y}).$$

Since $F(\boldsymbol{y}) \in \overline{\boldsymbol{Q}}[[\boldsymbol{y}]] \cap \overline{\boldsymbol{Q}}(\boldsymbol{y})$, there are $A(\boldsymbol{y}), B(\boldsymbol{y}) \in \overline{\boldsymbol{Q}}[\boldsymbol{y}]$ such that

$$F(\boldsymbol{y}) = A(\boldsymbol{y})/B(\boldsymbol{y}), \quad B(\boldsymbol{0}) \ne 0$$

(see Nishioka [9, Lemma 4]). Letting $\boldsymbol{y}_1 = \cdots = \boldsymbol{y}_s = \boldsymbol{z} = (z_1, \dots, z_n)$, we have

$$\begin{aligned} G(\boldsymbol{z}) &= F(\underbrace{\boldsymbol{z}, \dots, \boldsymbol{z}}_{s}) \\ &= \sum_{k \ge q} \sum_{m=1}^{M} \left(\sum_{\sigma=1}^{\nu} c_{m\sigma} \alpha_\sigma^k \right) \left(\frac{\zeta_1^{a_k} P(\Omega^k \boldsymbol{z})^{E_1}}{g_1 + \sum_{i=2}^{r} g_i \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}} \right)^m \\ &\in \overline{\boldsymbol{Q}}(z_1, \dots, z_n), \end{aligned}$$

where $E_i = \sum_{j=1}^{s} e_{ij} \in \boldsymbol{N}^+$ $(1 \le i \le r)$, since $e_{i1}, \dots, e_{is}$ are not all zero for each i. Letting $\sum_{\sigma=1}^{\nu} c_{m\sigma} \alpha_\sigma^k = d_m(k) \alpha_1^k$ $(1 \le m \le M)$, we find

$$d_m(k+p) = d_m(k) \quad (k \ge 0)$$

by (21). Then $G(\boldsymbol{z})$ satisfies the functional equation

$$G(\boldsymbol{z}) = \alpha_1^p G(\Omega^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{m=1}^{M} d_m(k) \alpha_1^k \left(\frac{\zeta_1^{a_k} P(\Omega^k \boldsymbol{z})^{E_1}}{g_1 + \sum_{i=2}^{r} g_i \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}} \right)^m,$$

so that by Theorem 1,

$$Q_k(X) = \sum_{m=1}^{M} d_m(k) \alpha_1^k \left(\frac{\zeta_1^{a_k} X^{E_1}}{g_1 + \sum_{i=2}^{r} g_i \zeta_i^{a_k} X^{E_i}} \right)^m \in \overline{\boldsymbol{Q}} \quad (q \le k \le p+q-1).$$

Hence

$$d_m(k)=0 \quad (1\le m\le M,\ q\le k\le p+q-1),$$

since $\mathrm{ord}_{X=0}\left(\zeta_1^{a_k}X^{E_1}/(g_1+\sum_{i=2}^{r}g_i\zeta_i^{a_k}X^{E_i})\right)^m = mE_1$ $(1\le m\le M)$. Letting $\eta_\sigma=\alpha_\sigma/\alpha_1$ $(1\le\sigma\le\nu)$, we see that $\eta_1,\cdots,\eta_\nu$ are distinct p-th roots of unity by (21) and that $d_m(k)=\sum_{\sigma=1}^{\nu}c_{m\sigma}\eta_\sigma^k=0$ $(q\le k\le p+q-1)$, which holds only if $c_{m1}=\cdots=c_{m\nu}=0$. This is a contradiction, since $c_{m\sigma}$ $(1\le m\le M,\ 1\le\sigma\le\nu)$ are not all zero, and the proof of the theorem is completed.

Proof of Theorem 4. We assume that

$$\sum_{k\ge 0}{}' \frac{k^l\alpha_\sigma^k}{(R_{a_k+h})^m} \qquad (1\le\sigma\le t,\ 0\le l\le L,\ -H\le h\le H,\ 1\le m\le M)$$

are algebraically dependent, where $\alpha_1,\ldots,\alpha_t$ are nonzero distinct algebraic numbers. Since $|\rho_1|>\max\{1,|\rho_2|\}$, there exists a nonnegative integer $q\ge\max\{k_0,k_1\}$ such that $R_{a_k+h}\ne 0$ for any h $(-H\le h\le H)$ and for all $k\ge q$, where k_0 and k_1 are as in the proof of Theorem 3. Define

$$f_{h,m}(x,\boldsymbol{y})=\sum_{k\ge q}x^k\left(\frac{\zeta_1^{a_k}\prod_{j=1}^{s}P(\Omega^k\boldsymbol{y}_j)^{e_{1j}}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h\zeta_2^{a_k}\prod_{j=1}^{s}P(\Omega^k\boldsymbol{y}_j)^{e_{2j}}}\right)^m \qquad (-H\le h\le H,\ 1\le m\le M),$$

where $P(\boldsymbol{z}),\Omega$ are given by (4), (6), respectively, and the roots of unity ζ_1,ζ_2 and the nonnegative integers e_{ij} $(i=1,2,\ 1\le j\le s)$ are determined by (18). Letting D and $\boldsymbol{\beta}$ be as in the proof of Theorem 3, we see that

$$\begin{aligned}(g_1^{-1}\rho_1^{-h})^mD^lf_{h,m}(\alpha_\sigma,\boldsymbol{\beta}) &= \sum_{k\ge q}k^l\alpha_\sigma^k\left(\frac{g_1^{-1}\rho_1^{-h}\rho_1^{-a_k}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h(\rho_1^{-1}\rho_2)^{a_k}}\right)^m\\ &= \sum_{k\ge q}\frac{k^l\alpha_\sigma^k}{(R_{a_k+h})^m}.\end{aligned}$$

Hence

$$\sum_{k\ge 0}{}'\frac{k^l\alpha_\sigma^k}{(R_{a_k+h})^m}-(g_1^{-1}\rho_1^{-h})^mD^lf_{h,m}(\alpha_\sigma,\boldsymbol{\beta})\in\overline{\boldsymbol{Q}}$$
$$(0\le l\le L,\ -H\le h\le H,\ 1\le m\le M,\ 1\le\sigma\le t),$$

and so $D^lf_{h,m}(\alpha_\sigma,\boldsymbol{\beta})$ $(0\le l\le L,\ -H\le h\le H,\ 1\le m\le M,\ 1\le\sigma\le t)$ are algebraically dependent. By the same way as in the proof of Theorem 3, we see that

$$\alpha_1^p=\cdots=\alpha_\nu^p \tag{22}$$

and $f_{h,m}(\alpha_\sigma,\boldsymbol{y})$ $(-H\le h\le H,\ 1\le m\le M,\ 1\le\sigma\le\nu)$ are linearly dependent over $\overline{\boldsymbol{Q}}$ modulo $\overline{\boldsymbol{Q}}(\boldsymbol{y})$, changing the indices σ $(1\le\sigma\le t)$ if necessary. Thus there are algebraic numbers $c_{hm\sigma}$ $(-H\le h\le H,\ 1\le m\le M,\ 1\le\sigma\le\nu)$, not all zero, such that

$$F(\boldsymbol{y}):=\sum_{h=-H}^{H}\sum_{m=1}^{M}\sum_{\sigma=1}^{\nu}c_{hm\sigma}f_{h,m}(\alpha_\sigma,\boldsymbol{y})\in\overline{\boldsymbol{Q}}(\boldsymbol{y}).$$

Letting $\boldsymbol{y}_1=\cdots=\boldsymbol{y}_s=\boldsymbol{z}=(z_1,\dots,z_n)$, we have

$$\begin{aligned}G(\boldsymbol{z})&=F(\underbrace{\boldsymbol{z},\dots,\boldsymbol{z}}_{s})\\&=\sum_{k\ge q}\sum_{h=-H}^{H}\sum_{m=1}^{M}\left(\sum_{\sigma=1}^{\nu}c_{hm\sigma}\alpha_\sigma^k\right)\left(\frac{\zeta_1^{a_k}P(\Omega^k\boldsymbol{z})^{E_1}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h\zeta_2^{a_k}P(\Omega^k\boldsymbol{z})^{E_2}}\right)^m\\&\in\overline{\boldsymbol{Q}}(z_1,\dots,z_n),\end{aligned}$$

where $E_i=\sum_{j=1}^{s}e_{ij}\in\boldsymbol{N}^+$ $(i=1,2)$, since $e_{i1},\dots,e_{is}$ are not all zero for each i. Letting $\sum_{\sigma=1}^{\nu}c_{hm\sigma}\alpha_\sigma^k=d_{hm}(k)\alpha_1^k$ $(-H\le h\le H,\ 1\le m\le M)$, we find

$$d_{hm}(k+p)=d_{hm}(k)\quad(k\ge 0)$$

by (22). Then $G(\boldsymbol{z})$ satisfies the functional equation

$$\begin{aligned}G(\boldsymbol{z})&=\alpha_1^pG(\Omega^p\boldsymbol{z})\\&\quad+\sum_{k=q}^{p+q-1}\sum_{h=-H}^{H}\sum_{m=1}^{M}d_{hm}(k)\alpha_1^k\left(\frac{\zeta_1^{a_k}P(\Omega^k\boldsymbol{z})^{E_1}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h\zeta_2^{a_k}P(\Omega^k\boldsymbol{z})^{E_2}}\right)^m,\end{aligned}$$

so that by Theorem 1,

$$\begin{aligned}Q_k(X)&=\sum_{h=-H}^{H}\sum_{m=1}^{M}d_{hm}(k)\alpha_1^k\left(\frac{\zeta_1^{a_k}X^{E_1}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h\zeta_2^{a_k}X^{E_2}}\right)^m\\&\in\overline{\boldsymbol{Q}}\quad(q\le k\le p+q-1).\end{aligned}$$

Hence

$$d_{hm}(k)=0\quad(-H\le h\le H,\ 1\le m\le M,\ q\le k\le p+q-1),$$

since $Q_k(X)$ has some poles if $d_{hm}(k)$ $(-H\le h\le H,\ 1\le m\le M)$ are not all zero. The rest of the proof is similar to that of Theorem 3.

Proof of Theorem 5. We assume that

$$\sum_{k\geq 0}{}' \frac{k^l\alpha_\sigma^k}{R_{ma_k+h}} \qquad (1\leq\sigma\leq t,\ 0\leq l\leq L,\ -H\leq h\leq H,\ 1\leq m\leq M)$$

are algebraically dependent, where $\alpha_1,\dots,\alpha_t$ are nonzero distinct algebraic numbers. Since $|\rho_1|>\max\{1,|\rho_2|\}$, there exists a nonnegative integer $q\geq\max\{k_0,k_1\}$ such that $R_{ma_k+h}\neq 0$ for any h $(-H\leq h\leq H)$, m $(1\leq m\leq M)$, and for all $k\geq q$, where k_0 and k_1 are as in the proof of Theorem 3. Define

$$f_{h,m}(x,\boldsymbol{y})=\sum_{k\geq q}\frac{x^k\zeta_1^{ma_k}\prod_{j=1}^s P(\Omega^k\boldsymbol{y}_j)^{me_{1j}}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h\zeta_2^{ma_k}\prod_{j=1}^s P(\Omega^k\boldsymbol{y}_j)^{me_{2j}}}$$
$$(-H\leq h\leq H,\ 1\leq m\leq M),$$

where $P(\boldsymbol{z}),\Omega$ are given by (4), (6), respectively, and the roots of unity ζ_1,ζ_2 and the nonnegative integers e_{ij} $(i=1,2,\ 1\leq j\leq s)$ are determined by (18). Letting D and $\boldsymbol{\beta}$ be as in the proof of Theorem 3, we see that

$$g_1^{-1}\rho_1^{-h}D^l f_{h,m}(\alpha_\sigma,\boldsymbol{\beta})=\sum_{k\geq q}\frac{k^l\alpha_\sigma^k g_1^{-1}\rho_1^{-h}\rho_1^{-ma_k}}{1+g_1^{-1}g_2(\rho_1^{-1}\rho_2)^h(\rho_1^{-1}\rho_2)^{ma_k}}=\sum_{k\geq q}\frac{k^l\alpha_\sigma^k}{R_{ma_k+h}}.$$

Hence

$$\sum_{k\geq 0}{}' \frac{k^l\alpha_\sigma^k}{R_{ma_k+h}}-g_1^{-1}\rho_1^{-h}D^l f_{h,m}(\alpha_\sigma,\boldsymbol{\beta})\in\overline{\boldsymbol{Q}}$$
$$(0\leq l\leq L,\ -H\leq h\leq H,\ 1\leq m\leq M,\ 1\leq\sigma\leq t),$$

and so $D^l f_{h,m}(\alpha_\sigma,\boldsymbol{\beta})$ $(0\leq l\leq L,\ -H\leq h\leq H,\ 1\leq m\leq M,\ 1\leq\sigma\leq t)$ are algebraically dependent. By the same way as in the proof of Theorem 3, we see that

$$\alpha_1^p=\cdots=\alpha_\nu^p \tag{23}$$

and $f_{h,m}(\alpha_\sigma,\boldsymbol{y})$ $(-H\leq h\leq H,\ 1\leq m\leq M,\ 1\leq\sigma\leq\nu)$ are linearly dependent over $\overline{\boldsymbol{Q}}$ modulo $\overline{\boldsymbol{Q}}(\boldsymbol{y})$, changing the indices σ $(1\leq\sigma\leq t)$ if necessary. Thus there are algebraic numbers $c_{hm\sigma}$ $(-H\leq h\leq H,\ 1\leq m\leq M,\ 1\leq\sigma\leq\nu)$, not all zero, such that

$$F(\boldsymbol{y}):=\sum_{h=-H}^{H}\sum_{m=1}^{M}\sum_{\sigma=1}^{\nu}c_{hm\sigma}f_{h,m}(\alpha_\sigma,\boldsymbol{y})\in\overline{\boldsymbol{Q}}(\boldsymbol{y}).$$

Letting $\boldsymbol{y}_1 = \cdots = \boldsymbol{y}_s = \boldsymbol{z} = (z_1, \ldots, z_n)$, we have

$$\begin{aligned} G(\boldsymbol{z}) &= F(\underbrace{\boldsymbol{z}, \ldots, \boldsymbol{z}}_{s}) \\ &= \sum_{k \ge q} \sum_{h=-H}^{H} \sum_{m=1}^{M} \frac{(\sum_{\sigma=1}^{\nu} c_{hm\sigma} \alpha_\sigma^k) \zeta_1^{m a_k} P(\Omega^k \boldsymbol{z})^{mE_1}}{1 + g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h \zeta_2^{m a_k} P(\Omega^k \boldsymbol{z})^{mE_2}} \\ &\in \overline{\boldsymbol{Q}}(z_1, \ldots, z_n), \end{aligned}$$

where $E_i = \sum_{j=1}^{s} e_{ij} \in \boldsymbol{N}^+$ $(i = 1, 2)$, since $e_{i1}, \ldots, e_{is}$ are not all zero for each i. Letting $\sum_{\sigma=1}^{\nu} c_{hm\sigma} \alpha_\sigma^k = d_{hm}(k) \alpha_1^k$ $(-H \le h \le H,\ 1 \le m \le M)$, we find

$$d_{hm}(k + p) = d_{hm}(k) \quad (k \ge 0)$$

by (23). Then $G(\boldsymbol{z})$ satisfies the functional equation

$$G(\boldsymbol{z}) = \alpha_1^p G(\Omega^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{h=-H}^{H} \sum_{m=1}^{M} \frac{d_{hm}(k) \alpha_1^k \zeta_1^{m a_k} P(\Omega^k \boldsymbol{z})^{mE_1}}{1 + g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h \zeta_2^{m a_k} P(\Omega^k \boldsymbol{z})^{mE_2}},$$

so that by Theorem 1,

$$\begin{aligned} Q_k(X) &= \sum_{h=-H}^{H} \sum_{m=1}^{M} \frac{d_{hm}(k) \alpha_1^k \zeta_1^{m a_k} X^{mE_1}}{1 + g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h \zeta_2^{m a_k} X^{mE_2}} \\ &= \sum_{m=1}^{M} \sum_{r=0}^{\infty} \left(\sum_{h=-H}^{H} d_{hm}(k) \alpha_1^k \zeta_1^{m a_k} \left(-g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h \zeta_2^{m a_k} \right)^r \right) X^{m(E_1 + E_2 r)} \\ &\in \overline{\boldsymbol{Q}} \quad (q \le k \le p + q - 1). \end{aligned}$$

We assert that $d_{hm}(k) = 0$ $(-H \le h \le H,\ 1 \le m \le M,\ q \le k \le p + q - 1)$. To the contrary we assume that $d_{hm}(k')$ $(-H \le h \le H,\ 1 \le m \le M)$ are not all zero for some k' $(q \le k' \le p + q - 1)$. Let

$$m' = \min\{\ m \mid d_{hm}(k')\ (-H \le h \le H) \text{ are not all zero }\}$$

and let $A = (E_1, E_2)$, $E_1' = E_1/A$, and $E_2' = E_2/A$. Then $(E_1', E_2') = 1$, and if $E_1' + E_2' r'$ $(r' \in \boldsymbol{N})$ is a prime number greater than M,

$$m'(E_1' + E_2' r') \ne m(E_1' + E_2' r)$$

for any m with $m' < m \le M$ and for all $r \ge 0$. Hence the linear recurrence

$$u_{m'k'}(r) = \sum_{h=-H}^{H} d_{hm'}(k') \alpha_1^{k'} \zeta_1^{m' a_{k'}} \left(-g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h \zeta_2^{m' a_{k'}} \right)^r = 0$$

for any $r \in \boldsymbol{N}$ such that $E_1' + E_2' r$ is a prime number greater than M. By Dirichlet's theorem on arithmetical progressions, there exist infinitely many such r. Therefore

$$d_{hm'}(k') = 0 \quad (-H \le h \le H) \tag{24}$$

by Skolem–Mahler–Lech's theorem (cf. Nishioka [11]), since $|g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^h| \ne |g_1^{-1} g_2 (\rho_1^{-1}\rho_2)^{h'}|$ if $h \ne h'$. However, (24) contradicts the choice of m', and so we can conclude that

$$d_{hm}(k) = 0 \ (-H \le h \le H,\ 1 \le m \le M,\ q \le k \le p+q-1).$$

The rest of the proof is similar to that of Theorem 3.

Proof of Theorem 6. There exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_s$ with $0 < |\beta_j| < 1$ $(1 \le j \le s)$ such that

$$\alpha_i = \zeta_i \prod_{j=1}^{s} \beta_j^{\,e_{ij}} \quad (1 \le i \le r), \tag{25}$$

where $\zeta_1, \dots, \zeta_r$ are roots of unity and e_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ are nonnegative integers. Take a positive integer N such that $\zeta_i^{\,N} = 1$ for any i $(1 \le i \le r)$. We can choose a positive integer p and a nonnegative integer q such that $a_{k+p} \equiv a_k \pmod N$ for any $k \ge q$. Let $y_{j\lambda}$ $(1 \le j \le s,\ 1 \le \lambda \le n)$ be variables and let $\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn})$ $(1 \le j \le s)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_s)$. Define

$$\begin{aligned} f_i(\boldsymbol{y}) &= \sum_{k \ge q} \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}, \\ g_i(\boldsymbol{y}) &= \sum_{k \ge q} \frac{\zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}, \end{aligned}$$

and

$$h_i(\boldsymbol{y}) = \prod_{k \ge q} \left(1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}\right) \qquad (1 \le i \le r),$$

where $P(\boldsymbol{z})$ and Ω are defined by (4) and (6), respectively. Letting

$$\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s),$$

we see that

$$f_i(\boldsymbol{\beta}) = \sum_{k\ge q} \alpha_i^{a_k}, \quad g_i(\boldsymbol{\beta}) = \sum_{k\ge q} \frac{\alpha_i^{a_k}}{1-\alpha_i^{a_k}}, \quad h_i(\boldsymbol{\beta}) = \prod_{k\ge q}(1-\alpha_i^{a_k}),$$

and so it suffices to prove the algebraic independence of the values $f_i(\boldsymbol{\beta}),\ g_i(\boldsymbol{\beta}),\ h_i(\boldsymbol{\beta})\ (1 \le i \le r)$. Let

$$\Omega' = \mathrm{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{s}).$$

Then $f_i(\boldsymbol{y}),\ g_i(\boldsymbol{y}),\ h_i(\boldsymbol{y})\ (1 \le i \le r)$ satisfy the functional equations

$$\begin{aligned} f_i(\boldsymbol{y}) &= f_i(\Omega'\boldsymbol{y}) + \sum_{k=q}^{p+q-1} \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}, \\ g_i(\boldsymbol{y}) &= g_i(\Omega'\boldsymbol{y}) + \sum_{k=q}^{p+q-1} \frac{\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}{1-\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}, \end{aligned}$$

and

$$h_i(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1}\left(1-\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}\right)\right) h_i(\Omega'\boldsymbol{y}),$$

where $\Omega'\boldsymbol{y} = (\Omega^p\boldsymbol{y}_1, \dots, \Omega^p\boldsymbol{y}_s)$. We assume that the values $f_i(\boldsymbol{\beta}),\ g_i(\boldsymbol{\beta}),\ h_i(\boldsymbol{\beta})$ $(1 \le i \le r)$ are algebraically dependent. Then the functions $f_i(\boldsymbol{y}),\ g_i(\boldsymbol{y}),\ h_i(\boldsymbol{y})$ $(1 \le i \le r)$ are algebraically dependent over $\overline{\boldsymbol{Q}}(\boldsymbol{y})$ by Lemmas 1 and 3. Hence by Lemma 5 at least one of the following two cases arises:

(i) There are algebraic numbers b_i, c_i $(1 \le i \le r)$, not all zero, and $F(\boldsymbol{y}) \in \overline{\boldsymbol{Q}}(\boldsymbol{y})$ such that

$$\begin{aligned} F(\boldsymbol{y}) &= F(\Omega'\boldsymbol{y}) \\ &+ \sum_{k=q}^{p+q-1}\sum_{i=1}^{r}\left(b_i\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}} + \frac{c_i\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}{1-\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}\right). \end{aligned} \tag{26}$$

(ii) There are rational integers d_i $(1 \le i \le r)$, not all zero, and $G(\boldsymbol{y}) \in \overline{\boldsymbol{Q}}(\boldsymbol{y}) \setminus \{0\}$ such that

$$G(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1}\prod_{i=1}^{r}\left(1-\zeta_i^{a_k}\prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}\right)^{d_i}\right) G(\Omega'\boldsymbol{y}). \tag{27}$$

Let M be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j}) \quad (1 \le j \le s),$$

where M is so large that the following two properties are both satisfied:

(A) If $(e_{i1}, \dots, e_{is}) \ne (e_{i'1}, \dots, e_{i's})$, then $\sum_{j=1}^{s} e_{ij} M^j \ne \sum_{j=1}^{s} e_{i'j} M^j$.

(B) $F^*(\boldsymbol{z}) = F(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \overline{\boldsymbol{Q}}(z_1, \dots, z_n)$,
$G^*(\boldsymbol{z}) = G(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \overline{\boldsymbol{Q}}(z_1, \dots, z_n) \setminus \{0\}$.

Then by (26) and (27), at least one of the following two functional equations holds:

$$F^*(\boldsymbol{z}) = F^*(\Omega^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{i=1}^{r} \left(b_i \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i} + \frac{c_i \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}}{1 - \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}} \right), \tag{28}$$

$$G^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \left(1 - \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i} \right)^{d_i} \right) G(\Omega^p \boldsymbol{z}), \tag{29}$$

where $E_i = \sum_{j=1}^{s} e_{ij} M^j$ $(1 \le i \le r)$ are distinct positive integers by the property (A), since none of α_i/α_j $(1 \le i < j \le r)$ is a root of unity. By Theorems 1, 2, and the property (B), at least one of the following two properties is satisfied:

(i) For any k $(q \le k \le p+q-1)$,

$$\sum_{i=1}^{r} \left(b_i \zeta_i^{a_k} X^{E_i} + \frac{c_i \zeta_i^{a_k} X^{E_i}}{1 - \zeta_i^{a_k} X^{E_i}} \right) = \sum_{i=1}^{r} \left(b_i \zeta_i^{a_k} X^{E_i} + c_i \sum_{l=1}^{\infty} (\zeta_i^{a_k} X^{E_i})^l \right) \in \overline{\boldsymbol{Q}}. \tag{30}$$

(ii) For any k $(q \le k \le p+q-1)$,

$$\prod_{i=1}^{r} (1 - \zeta_i^{a_k} X^{E_i})^{d_i} = \gamma_k \in \overline{\boldsymbol{Q}}^{\times}. \tag{31}$$

Suppose first that (28) is satisfied. Then we show that $c_i = 0$ $(1 \le i \le r)$. Assume contrary that $c_1, \dots, c_r$ are not all zero. Let $S = \{\, i \in \{1, \dots, r\} \mid c_i \ne 0 \,\}$ and let $i' \in S$ be the index such that $E_{i'} < E_i$ for any $i \in S \setminus \{i'\}$. Since $(E_1 \cdots E_r + 1) E_{i'}$ is not divided by any E_i with $i \in S \setminus \{i'\}$, the term $c_{i'} (\zeta_{i'}^{a_k} X^{E_{i'}})^{E_1 \cdots E_r + 1}$ does not cancel in (30), which is a contradiction. Hence

$c_i = 0$ $(1 \le i \le r)$ and so $b_1, \ldots, b_r$ are not all zero, which is also a contradiction, since $E_1, \ldots, E_r$ are distinct. Next suppose that (29) is satisfied. Taking the logarithmic derivative of (31), we get

$$\sum_{i=1}^{r} \frac{-d_i E_i \zeta_i^{a_k} X^{E_i - 1}}{1 - \zeta_i^{a_k} X^{E_i}} = 0 \quad (q \le k \le p + q - 1).$$

This is a contradiction, since $\mathrm{ord}_{X=0} E_i \zeta_i^{a_k} X^{E_i - 1}/(1 - \zeta_i^{a_k} X^{E_i}) = E_i - 1$ $(1 \le i \le r)$, and the proof of the theorem is completed.

3 Proofs of Theorems 1 and 2.

We need several lemmas to prove Theorems 1 and 2. Use the same notations as in the preceding section, define an endomorphism $\tau : M \to M$ by (17), and adopt the usual vector notation, that is, if $I = (i_1, \ldots, i_n) \in \boldsymbol{Z}^n$, we write $\boldsymbol{z}^I = z_1^{i_1} \cdots z_n^{i_n}$. We denote by $C[z_1, \ldots, z_n]$ the ring of polynomials in variables $z_1, \ldots, z_n$ with coefficients in C.

Lemma 6 (Nishioka [11]). *If $A, B \in C[z_1, \ldots, z_n]$ are coprime, then $(A^\tau, B^\tau) = \boldsymbol{z}^I$, where $I \in \boldsymbol{N}^n$.*

Lemma 7 (Nishioka [10], cf. [11]). *Let Ω be an $n \times n$ matrix with nonnegative integer entries which has the property* (I). *Let $\overline{C}$ be an algebraically closed field of characteristic* 0. *Let $R(\boldsymbol{z})$ be a nonzero polynomial in $\overline{C}[z_1, \ldots, z_n]$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$ an element of $\overline{C}^n$ with $x_i \neq 0$ for any i $(1 \le i \le n)$. We put*

$$R(\boldsymbol{z}) = \sum_{I = (i_1, \ldots, i_n) \in \Lambda} c_I \boldsymbol{z}^I \quad (c_I \neq 0).$$

If $R(\Omega^k \boldsymbol{x}) = 0$ for infinitely many positive integers k, then there exist distinct elements $I, J \in \Lambda$ and positive integers a, b such that

$$\boldsymbol{x}^{(I-J)\Omega^a(\Omega^{bk} - E)} = 1$$

for all $k \ge 0$, where E is the identity matrix.

Lemma 8 (Nishioka [9]). *If $g \in M$ satisfies*

$$g^\tau = cg + d \quad (c, d \in C),$$

then $g \in C$.

Lemma 9. *Let* $\{a_k\}_{k\geq 0}$ *be a linear recurrence satisfying* (2)*. Suppose that* $\{a_k\}_{k\geq 0}$ *is not a geometric progression. Assume that the ratio of any pair of distinct roots of* $\Phi(X)$ *is not a root of unity, where* $\Phi(X)$ *is the polynomial defined by* (3)*. Then the sequence* $\{a_k\}_{k\geq 0}$ *does not satisfy the linear recurrence relation of the form*

$$a_{k+l} \;=\; ca_k \quad (k \geq 0),$$

where l *is a positive integer and* c *is a nonzero rational number.*

Proof. If $l = 1$, then $a_k = a_0 c^k$ $(k \geq 0)$, which contradicts the assumption in the lemma. If $l \geq 2$, then at least two of the roots of $\Psi(X) = X^l - c$ are those of $\Phi(X)$. This also contradicts the assumption, since the ratio of any pair of distinct roots of $\Psi(X)$ is a root of unity.

Lemma 10. *Let* $\boldsymbol{u} = (u_1, \dots, u_n)$ *satisfy* $\operatorname{trans.deg}_C C(\boldsymbol{u}) = n - 1$*. If* $\boldsymbol{u}^I, \boldsymbol{u}^J \in C^\times$*, where* $I, J \in \boldsymbol{Z}^n \setminus \{\boldsymbol{0}\}$*, then* I *and* J *are proportional, i.e., there exists a nonzero rational number* r *such that* $I = rJ$*.*

Proof. Suppose contrary there are $I = (i_1, \dots, i_n), J = (j_1, \dots, j_n) \in \boldsymbol{Z}^n \setminus \{\boldsymbol{0}\}$ such that $\boldsymbol{u}^I, \boldsymbol{u}^J \in C^\times$ and I, J are not proportional. Assume that $j_\lambda \neq 0$. Then u_λ is algebraic over the field $C(u_1, \dots, u_{\lambda-1}, u_{\lambda+1}, \dots, u_n)$. Since $(\boldsymbol{u}^I)^{j_\lambda}(\boldsymbol{u}^J)^{-i_\lambda} = \boldsymbol{u}^{j_\lambda I - i_\lambda J} \in C^\times$ and $j_\lambda I - i_\lambda J$ is a nonzero vector whose λ-th component is zero, $u_1, \dots, u_{\lambda-1}, u_{\lambda+1}, \dots, u_n$ are algebraically dependent over C. Hence $\operatorname{trans.deg}_C C(\boldsymbol{u}) \leq n - 2$, which is a contradiction.

Lemma 11. *Let* $\{a_k\}_{k\geq 0}$ *be as in Lemma* 9*. If* $k_1, k_2 \in \boldsymbol{N}$ *are distinct, then* $P(\Omega^{k_1}\boldsymbol{z}) - \gamma_1$ *and* $P(\Omega^{k_2}\boldsymbol{z}) - \gamma_2$ *are coprime, where* $P(\boldsymbol{z})$ *is the monomial defined by* (4)*,* Ω *is the matrix defined by* (6)*, and* $\gamma_1, \gamma_2 \in C^\times$*.*

Proof. Suppose contrary there is an irreducible $T(\boldsymbol{z}) \in C[z_1, \dots, z_n] \setminus C$ which divides both $P(\Omega^{k_1}\boldsymbol{z}) - \gamma_1$ and $P(\Omega^{k_2}\boldsymbol{z}) - \gamma_2$. We may assume that $k_1 > k_2$. Let $\boldsymbol{u} = (u_1, \dots, u_n)$ be a generic point of the algebraic variety defined by $T(\boldsymbol{z})$ over C. Then $T(\boldsymbol{u}) = 0$ and $\operatorname{trans.deg}_C C(\boldsymbol{u}) = n - 1$. Since $T(\boldsymbol{u}) = 0$,

$$P(\Omega^{k_1}\boldsymbol{u}) = u_1^{a_{k_1+n-1}} \cdots u_n^{a_{k_1}} = \gamma_1$$

and

$$P(\Omega^{k_2}\boldsymbol{u}) = u_1^{a_{k_2+n-1}} \cdots u_n^{a_{k_2}} = \gamma_2.$$

By Lemma 10, there exists a nonzero rational number c such that $(a_{k_1+n-1}, \dots, a_{k_1}) = c(a_{k_2+n-1}, \dots, a_{k_2})$. Hence by (2), $\{a_k\}_{k\geq 0}$ satisfies the linear recurrence relation $a_{k+k_1-k_2} = ca_k$ $(k \geq 0)$, which contradicts Lemma 9.

Lemma 12. *Let Ω be an $n \times n$ matrix with nonnegative integer entries which has the property* (I). *Let $R(\boldsymbol{z})$ be a nonzero polynomial in $C[z_1, \dots, z_n]$. If $R(\Omega \boldsymbol{z})$ divides $R(\boldsymbol{z})\boldsymbol{z}^I$, where $I \in \boldsymbol{N}^n$, then $R(\boldsymbol{z})$ is a monomial in $z_1, \dots, z_n$.*

Proof. We can put

$$R(\boldsymbol{z}) = \boldsymbol{z}^J \prod_{i=1}^{\nu} g_i(\boldsymbol{z})^{e_i},$$

where $J \in \boldsymbol{N}^n$, e_i $(1 \leq i \leq \nu)$ are positive integers, and $g_1(\boldsymbol{z}), \dots, g_\nu(\boldsymbol{z})$ are distinct irreducible polynomials and not monomials. For each i $(1 \leq i \leq \nu)$, $g_i(\Omega \boldsymbol{z})$ can be written as

$$g_i(\Omega \boldsymbol{z}) = h_i(\boldsymbol{z})\boldsymbol{z}^{H_i},$$

where $h_i(\boldsymbol{z}) \in C[z_1, \dots, z_n] \setminus C$ is not divided by $z_1, \dots, z_n$, and $H_i \in \boldsymbol{N}^n$. Since $\boldsymbol{z}^{J\Omega} \prod_{i=1}^{\nu} (h_i(\boldsymbol{z})\boldsymbol{z}^{H_i})^{e_i}$ divides $\boldsymbol{z}^{I+J} \prod_{i=1}^{\nu} g_i(\boldsymbol{z})^{e_i}$,

$$\prod_{i=1}^{\nu} h_i(\boldsymbol{z})^{e_i} \ \Big| \ \prod_{i=1}^{\nu} g_i(\boldsymbol{z})^{e_i}. \tag{32}$$

Hence $h_1(\boldsymbol{z}), \dots, h_\nu(\boldsymbol{z})$ are irreducible, otherwise we can deduce a contradiction, comparing the numbers of prime factors in (32); thereby

$$\prod_{i=1}^{\nu} h_i(\boldsymbol{z})^{e_i} = \xi \prod_{i=1}^{\nu} g_i(\boldsymbol{z})^{e_i},$$

where ξ is a nonzero element of C. Therefore

$$R(\Omega \boldsymbol{z}) = \xi R(\boldsymbol{z})\boldsymbol{z}^H, \quad H = J(\Omega - E) + \sum_{i=1}^{\nu} e_i H_i \in \boldsymbol{Z}^n.$$

Let $D = |\det(\Omega - E)|$. Then D is a positive integer, since the matrix Ω has no roots of unity as its eigenvalues. We extend the endomorphism $\tau : M \to M$ to the quotient field M' of formal power series ring $C[[z_1^{1/D}, \dots, z_n^{1/D}]]$ by the usual way. Since the monomial $S(\boldsymbol{z}) = \boldsymbol{z}^{H(\Omega-E)^{-1}} \in M'$ satisfies $S^\tau(\boldsymbol{z}) = S(\boldsymbol{z})\boldsymbol{z}^H$, we see that $F(\boldsymbol{z}) = R(\boldsymbol{z})/S(\boldsymbol{z}) \in M'$ satisfies $F^\tau(\boldsymbol{z}) = \xi F(\boldsymbol{z})$ and so $F(\boldsymbol{z}) \in C$ by Lemma 8, which means that $R(\boldsymbol{z})$ is a monomial in $z_1, \dots, z_n$.

Proof of Theorem 1. Letting $G(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $C[z_1, \dots, z_n]$, and letting for each k ($q \le k \le p+q-1$), $Q_k(X) = U_k(X)/V_k(X)$, where $U_k(X)$ and $V_k(X)$ are coprime polynomials in $C[X]$, we have

$$\begin{aligned} & A(\boldsymbol{z})B(\Omega^p\boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^k\boldsymbol{z})) \\ = \ & \alpha A(\Omega^p\boldsymbol{z})B(\boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^k\boldsymbol{z})) + R(\boldsymbol{z})B(\boldsymbol{z})B(\Omega^p\boldsymbol{z}), \end{aligned}$$

$$R(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} V_k(P(\Omega^k\boldsymbol{z})) \right) \sum_{k=q}^{p+q-1} Q_k(P(\Omega^k\boldsymbol{z})) \in C[z_1, \dots, z_n],$$

by (7). We can put $(A(\Omega^p\boldsymbol{z}), B(\Omega^p\boldsymbol{z})) = \boldsymbol{z}^I$, where $I \in \boldsymbol{N}^n$, by Lemma 6. Then

$$B(\Omega^p\boldsymbol{z}) \;\Big|\; B(\boldsymbol{z})\boldsymbol{z}^I \prod_{k=q}^{p+q-1} V_k(P(\Omega^k\boldsymbol{z})) \tag{33}$$

and

$$B(\boldsymbol{z}) \;\Big|\; B(\Omega^p\boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^k\boldsymbol{z})). \tag{34}$$

Let $\overline{C}$ be the algebraic closure of C. First we prove that $G(\boldsymbol{z}) \in C[z_1, \dots, z_n]$. For this purpose, we show that $B(\Omega^p\boldsymbol{z})$ divides $B(\boldsymbol{z})\boldsymbol{z}^I$. Otherwise, by (33), there exists a prime factor $T(\boldsymbol{z}) \in \overline{C}[z_1, \dots, z_n]$ of $B(\Omega^p\boldsymbol{z})$ such that

$$T(\boldsymbol{z}) \;\Big|\; (P(\Omega^{k_0}\boldsymbol{z}) - \gamma) \tag{35}$$

for some k_0 ($q \le k_0 \le p+q-1$) and a root γ of $V_{k_0}(X)$, so that γ is a nonzero element of $\overline{C}$, since $V_k(0) \ne 0$ ($q \le k \le p+q-1$) and so $V_{k_0}(0) \ne 0$. Let $\boldsymbol{u} = (u_1, \dots, u_n)$ be a generic point of the algebraic variety defined by $T(\boldsymbol{z})$ over $\overline{C}$. Then $T(\boldsymbol{u}) = 0$ and

$$\text{trans.}\deg_{\overline{C}} \overline{C}(\boldsymbol{u}) = n - 1.$$

Letting $\boldsymbol{z} = \boldsymbol{u}$ in (35), we see that

$$P(\Omega^{k_0}\boldsymbol{u}) = u_1^{a_{k_0+n-1}} \cdots u_n^{a_{k_0}} = \gamma. \tag{36}$$

Since $T(\boldsymbol{z})$ divides $B(\Omega^p \boldsymbol{z})$ and $B(\Omega^p \boldsymbol{z})$ divides $B(\Omega^{2p}\boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^{k+p}\boldsymbol{z}))$ by (34),

$$T(\boldsymbol{z}) \Bigm| B(\Omega^{2p}\boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^{k+p}\boldsymbol{z})).$$

Therefore $T(\boldsymbol{z})$ divides $B(\Omega^{2p}\boldsymbol{z})$ by Lemma 11 with (35). Continuing this process, we see that $T(\boldsymbol{z})$ divides $B(\Omega^{pk}\boldsymbol{z})$ and so $B(\Omega^{pk}\boldsymbol{u}) = 0$ for all positive integers k. Since $u_\lambda \neq 0$ $(1 \leq \lambda \leq n)$, by Lemmas 1 and 7, there exist a nonzero n-dimensional vector $\boldsymbol{v}$ with rational integer components and positive integers d, e such that $\boldsymbol{u}^{\boldsymbol{v}\Omega^e(\Omega^{dk}-E)} = 1$ for all $k \geq 0$, where E is the identity matrix. Then

$$\boldsymbol{u}^{\boldsymbol{v}(\Omega^d - E)\Omega^{dk+e}} = 1$$

for all $k \geq 0$. Letting $\boldsymbol{v}(\Omega^d - E)\Omega^e = (b_{n-1}, \ldots, b_0)$ and letting $\{b_k\}_{k\geq 0}$ be a linear recurrence defined by (2) with the initial values $b_0, \ldots, b_{n-1}$, we have

$$u_1^{b_{dk+n-1}} \cdots u_n^{b_{dk}} = 1 \tag{37}$$

for all $k \geq 0$. Therefore by Lemma 10, together with (2), $\{b_k\}_{k\geq 0}$ satisfies the linear recurrence relation

$$b_{k+d} = c b_k \quad (k \geq 0), \tag{38}$$

where c is a nonzero rational number. On the other hand, there exists a nonzero rational number c' such that $(a_{k_0+n-1}, \ldots, a_{k_0}) = c'(b_{n-1}, \ldots, b_0)$ by (36), (37), and Lemma 10. Hence by (2), we have

$$a_{k+k_0} = c' b_k \quad (k \geq 0). \tag{39}$$

By (38) and (39), $a_{k+d} = c a_k$ for all $k \geq k_0$. Then by (2), $a_{k+d} = c a_k$ $(k \geq 0)$, which contradicts Lemma 9, and so we can conclude that $B(\Omega^p \boldsymbol{z})$ divides $B(\boldsymbol{z})\boldsymbol{z}^I$. Therefore $B(\boldsymbol{z})$ is a monomial in $z_1, \ldots, z_n$ by Lemmas 1 and 12. Hence we can conclude that $G(\boldsymbol{z}) \in C[z_1, \ldots, z_n]$, since $G(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z}) \in C[[z_1, \ldots, z_n]]$.

Secondly we show that $Q_k(X) = U_k(X)/V_k(X) \in C[X]$ $(q \leq k \leq p+q-1)$. Since $U_k(X)$ and $V_k(X)$ are coprime in $C[X]$ with $V_k(0) \neq 0$, $U_k(P(\Omega^k \boldsymbol{z}))$ and $V_k(P(\Omega^k \boldsymbol{z}))$ are coprime polynomials in $C[z_1, \ldots, z_n]$ with $V_k(P(\mathbf{0})) \neq 0$. By Lemma 11, $V_k(P(\Omega^k \boldsymbol{z}))$ and $V_{k'}(P(\Omega^{k'} \boldsymbol{z}))$ are coprime if $k \neq k'$. Since

$G(\boldsymbol{z}) \in C[z_1, \ldots, z_n]$ and so $G(\Omega^p \boldsymbol{z}) \in C[z_1, \ldots, z_n]$,

$$\sum_{k=q}^{p+q-1} \frac{U_k(P(\Omega^k \boldsymbol{z}))}{V_k(P(\Omega^k \boldsymbol{z}))} \in C[z_1, \ldots, z_n]$$

by (7). Hence $V_k(P(\Omega^k \boldsymbol{z}))$ divides $U_k(P(\Omega^k \boldsymbol{z}))$ and so $V_k(P(\Omega^k \boldsymbol{z})) \in C^\times$ for any k ($q \le k \le p+q-1$). Therefore $V_k(X) \in C^\times$ and so $Q_k(X) \in C[X]$ ($q \le k \le p+q-1$).

Finally we prove that $Q_k(X) \in C$ ($q \le k \le p+q-1$), which implies $G(\boldsymbol{z}) \in C$ by Lemma 8. To the contrary we assume that $Q_k(X) \notin C$ for some k ($q \le k \le p+q-1$). Let g be the number of terms appearing in $G(\boldsymbol{z})$. Iterating (7), we get

$$G(\boldsymbol{z}) - \alpha^{2g+1} G(\Omega^{(2g+1)p} \boldsymbol{z}) = \sum_{l=0}^{2g} \alpha^l \sum_{k=q}^{p+q-1} Q_k(P(\Omega^{k+lp} \boldsymbol{z})).$$

Then the number of terms appearing in the right-hand side is at least $2g+1$, since $(a_{k+n-1} : \ldots : a_k) \ne (a_{k'+n-1} : \ldots : a_{k'})$ in $\boldsymbol{P}^{n-1}(\boldsymbol{Q})$ for any distinct non-negative integers k and k' by Lemma 9 and so the nonconstant terms appearing in the right-hand side never cancel one another. This is a contradiction, since the number of terms appearing in the left-hand side is at most $2g$, and the proof of the theorem is completed.

Proof of Theorem 2. Letting $G(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $C[z_1, \ldots, z_n]$, and letting for each k ($q \le k \le p+q-1$), $Q_k(X) = U_k(X)/V_k(X)$, where $U_k(X)$ and $V_k(X)$ are coprime polynomials in $C[X]$, we have

$$A(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^k \boldsymbol{z})) = A(\Omega^p \boldsymbol{z}) B(\boldsymbol{z}) \prod_{k=q}^{p+q-1} U_k(P(\Omega^k \boldsymbol{z})) \tag{40}$$

by (8). We can put $(A(\Omega^p \boldsymbol{z}), B(\Omega^p \boldsymbol{z})) = \boldsymbol{z}^I$, where $I \in \boldsymbol{N}^n$, by Lemma 6. Then

$$A(\Omega^p \boldsymbol{z}) \ \Big| \ A(\boldsymbol{z}) \boldsymbol{z}^I \prod_{k=q}^{p+q-1} V_k(P(\Omega^k \boldsymbol{z})),$$

$$A(\boldsymbol{z}) \ \Big| \ A(\Omega^p \boldsymbol{z}) \prod_{k=q}^{p+q-1} U_k(P(\Omega^k \boldsymbol{z})),$$

$$B(\Omega^p \boldsymbol{z}) \ \Big| \ B(\boldsymbol{z}) \boldsymbol{z}^I \prod_{k=q}^{p+q-1} U_k(P(\Omega^k \boldsymbol{z})),$$

and

$$B(\boldsymbol{z}) \;\Big|\; B(\Omega^p \boldsymbol{z}) \prod_{k=q}^{p+q-1} V_k(P(\Omega^k \boldsymbol{z})).$$

Since $U_k(0) \neq 0, V_k(0) \neq 0$ $(q \leq k \leq p+q-1)$, by the same way as in the proof of Theorem 1, we see that $A(\Omega^p \boldsymbol{z})$ divides $A(\boldsymbol{z})\boldsymbol{z}^I$ and that $B(\Omega^p \boldsymbol{z})$ divides $B(\boldsymbol{z})\boldsymbol{z}^I$. Therefore $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are monomials in $z_1, \ldots, z_n$ by Lemmas 1 and 12. Then by (40),

$$\prod_{k=q}^{p+q-1} U_k(P(\Omega^k \boldsymbol{z})) \Big/ \prod_{k=q}^{p+q-1} V_k(P(\Omega^k \boldsymbol{z})) \in C^\times.$$

Here, $U_k(P(\Omega^k \boldsymbol{z}))$ and $V_{k'}(P(\Omega^{k'} \boldsymbol{z}))$ $(k \neq k')$ are coprime polynomials in $C[z_1, \ldots, z_n]$ by Lemma 11, and $U_k(P(\Omega^k \boldsymbol{z})), V_k(P(\Omega^k \boldsymbol{z}))$ are coprime polynomials in $C[z_1, \ldots, z_n]$ for each k $(q \leq k \leq p+q-1)$. Therefore $U_k(X), V_k(X) \in C^\times$ $(q \leq k \leq p+q-1)$ and so $Q_k(X) \in C^\times$ $(q \leq k \leq p+q-1)$. Hence $G(\boldsymbol{z}) \in C$ by Lemma 8, and the proof of the theorem is completed.

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